Operations on 3D Rotations

C. Dario Bellicoso and Jan Carius

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Abstract

We compile a cheat sheet on addition and subtraction operations in the space of 3D orientations. We place special emphasis on denoting all reference frames of the involved quantities by a left subscript. The mathematical foundations are summarized in the paper *A Primer on the Differential Calculus of 3D Orientations* by Blösch et al. (`https://arxiv.org/abs/1606.05285`).

1 Notation and Basic Properties

- Rotation vectors $\in \mathbb{R}^3$

$$ {}_A\boldsymbol{\varphi}_{AB} = -{}_A\boldsymbol{\varphi}_{BA} \tag{1}$$

- Change of base coordinates (=applying a passive rotation)

$$\begin{aligned} {}_A\mathbf{v} &= \Phi_{AB}({}_B\mathbf{v}) \\ {}_B\mathbf{v} &= \Phi_{AB}^{-1}({}_A\mathbf{v}) \end{aligned} \tag{2}$$

- Exponential map of a rotation vector gives a rotation

$$\exp\left({}_A\boldsymbol{\varphi}_{AB}\right) = \exp\left({}_B\boldsymbol{\varphi}_{AB}\right) = \Phi_{AB} \tag{3}$$

$$\exp\left({}_A\boldsymbol{\varphi}_{AB}\right) = \Phi_{AB} = \Phi_{BA}^{-1} = \exp\left(-{}_B\boldsymbol{\varphi}_{BA}\right) = \exp\left({}_B\boldsymbol{\varphi}_{AB}\right) \tag{4}$$

- Definition of angular velocity:

$$ {}_B\boldsymbol{\omega}_{IB} = -\lim_{\epsilon\to 0}\frac{{}_B\boldsymbol{\varphi}_{B^+,B}}{\epsilon} \tag{5}$$

2 Rotation operations: the $\boxplus$ and $\boxminus$ operators

- Definition of $\boxplus$ and $\boxminus$

$$\begin{aligned} \boxplus :&\mathrm{SO}(3)\times\mathbb{R}^3 \to \mathrm{SO}(3) \\ &\Phi, \boldsymbol{\varphi} \to \exp\left(\boldsymbol{\varphi}\right)\otimes\Phi \\ \boxminus :&\mathrm{SO}(3)\times\mathrm{SO}(3) \to \mathbb{R}^3 \\ &\Phi_1, \Phi_2 \to \log\left(\Phi_1\otimes\Phi_2^{-1}\right) \end{aligned} \tag{6}$$

  Here, $\otimes$ means concatenation of rotations.

- Using $\boxplus$ with passive rotations

$$\begin{aligned} \Phi_{IB} &= \Phi_{IA} \boxplus {}_I\boldsymbol{\varphi}_{AB} \\ &= \Phi_{IA} \boxplus \Phi_{IA}({}_A\boldsymbol{\varphi}_{AB}) \\ &= \exp\left(\Phi_{IA}({}_A\boldsymbol{\varphi}_{AB})\right)\otimes\Phi_{IA} \\ &= \Phi_{IA}\otimes\exp\left(({}_A\boldsymbol{\varphi}_{AB})\right)\otimes\underbrace{\Phi_{IA}^{-1}\otimes\Phi_{IA}}_{=\Phi_I} \\ &= \Phi_{IA}\otimes\Phi_{AB} \end{aligned} \tag{7}$$

- Using $\boxplus$ with active rotations

$$
\begin{aligned}
\Phi_{BI} &= \Phi_{AI} \boxplus_A \boldsymbol{\varphi}_{BA} \\
&= \exp\left({}_A\boldsymbol{\varphi}_{BA}\right) \otimes \Phi_{AI} \\
&= \Phi_{BA} \otimes \Phi_{AI}
\end{aligned} \tag{8}
$$

- Using $\boxminus$ with active rotations

$$
\begin{aligned}
\Phi_{AI} \boxminus \Phi_{BI} &= \log\left(\Phi_{AI} \otimes \Phi_{BI}^{-1}\right) \\
&= \log\left(\Phi_{AB}\right) \\
&=_A \boldsymbol{\varphi}_{AB} \\
&=_B \boldsymbol{\varphi}_{AB}
\end{aligned} \tag{9}
$$

In short

$$
\Phi_{AI} \boxminus \Phi_{BI} = {}_B\boldsymbol{\varphi}_{AB} \tag{10}
$$

- Using $\boxminus$ with passive rotations

$$
\begin{aligned}
\Phi_{IA} \boxminus \Phi_{IB} &= \log\left(\Phi_{IA} \otimes \Phi_{IB}^{-1}\right) \\
&= \log\left(\Phi_{I} \otimes \Phi_{IA} \otimes \Phi_{IB}^{-1}\right) \\
&= \log\left(\Phi_{IB} \otimes \Phi_{IB}^{-1} \otimes \Phi_{IA} \otimes \Phi_{IB}^{-1}\right) \\
&= \log\left(\Phi_{IB} \otimes \Phi_{BA} \otimes \Phi_{IB}^{-1}\right) \\
&= \log\left(\Phi_{IB} \otimes \exp\left({}_B\boldsymbol{\varphi}_{BA}\right) \otimes \Phi_{IB}^{-1}\right) \\
&= \log\left(\exp\left(\Phi_{IB}({}_B\boldsymbol{\varphi}_{BA})\right)\right) \\
&= \Phi_{IB}({}_B\boldsymbol{\varphi}_{BA}) \\
&= {}_I\boldsymbol{\varphi}_{BA}
\end{aligned} \tag{11}
$$

In short, the result is

$$
\Phi_{IA} \boxminus \Phi_{IB} = {}_I\boldsymbol{\varphi}_{BA} \tag{12}
$$

- Updating the rotation of a body (passive rotation) with an incremental absolute angular velocity. The next state is denoted by prime:

$$
\begin{aligned}
\Phi_{IB'} &= \Phi_{IB} \boxplus \left({}_I\omega_{IB}\Delta t\right) \qquad (13) \\
&= \Phi_{IB} \boxplus \left(\Phi_{IB}(\underbrace{{}_B\omega_{IB}\Delta t}_{{}_B\boldsymbol{\varphi}_{BB'}})\right) \\
&= \exp\left(\Phi_{IB}({}_B\boldsymbol{\varphi}_{BB'})\right) \otimes \Phi_{IB} \\
&= \Phi_{IB} \exp\left({}_B\boldsymbol{\varphi}_{BB'}\right)\Phi_{IB}^{-1} \otimes \Phi_{IB} \\
&= \Phi_{IB} \otimes \Phi_{BB'}\ . \qquad (14)
\end{aligned}
$$

Same rule for active rotations

$$
\Phi_{B'I} = \Phi_{BI} \boxplus \left(-{}_B\omega_{IB}\Delta t\right) \tag{15}
$$

- Algebraic operations with boxminus and boxplus:

$$
\begin{aligned}
& \Phi_{IB} = \Phi_{IA} \boxplus {}_I\boldsymbol{\varphi}_{AB} \qquad (16) \\
\Leftrightarrow\ & \Phi_{IB} \boxminus \Phi_{IA} = {}_I\boldsymbol{\varphi}_{AB} \qquad (17)
\end{aligned}
$$

and

$$
\begin{aligned}
& \Phi_{BI} = \Phi_{AI} \boxplus {}_A\boldsymbol{\varphi}_{BA} \qquad (18) \\
\Leftrightarrow\ & \Phi_{BI} \boxminus \Phi_{AI} = {}_A\boldsymbol{\varphi}_{BA} \qquad (19)
\end{aligned}
$$

3 Identities

- Changing the frame of a rotation vector inside the exponential map

$$\exp\left(\Phi(\boldsymbol{\varphi})\right) = \Phi \otimes \exp\left(\boldsymbol{\varphi}\right) \otimes \Phi^{-1} \tag{20}$$

- Relationship between boxminus used with active and passive rotations. The definition of boxminus is unchanged.

$$\Phi_{AI} \boxminus \Phi_{BI} = -\Phi_{BI} \otimes (\Phi_{IA} \boxminus \Phi_{IB}) \tag{21}$$

- Reversing the order of the operands on boxminus

$$\Phi_{AI} \boxminus \Phi_{BI} = -(\Phi_{BI} \boxminus \Phi_{AI}) \tag{22}$$

- "Transforming" rotation matrices into a different frame by treating them as operators (useful for some proofs):

$$\color{red}{R_{IA}} R_{AB} \color{red}{R_{IA}^\top} = \color{red}{R_{IA}} R_{AI} R_{IB} \color{red}{R_{IA}^\top} = R_{IB} R_{AI} \tag{23}$$
$$\color{red}{R_{IB}} R_{BA} \color{red}{R_{IB}^\top} = \color{red}{R_{IB}} R_{BI} R_{IA} \color{red}{R_{IB}^\top} = R_{IA} R_{BI} \tag{24}$$

- Exponential map of a boxminus operation

$$\begin{aligned} \exp(\Phi_{AI} \boxminus \Phi_{BI}) &= \exp(\log(\Phi_{AI} \otimes \Phi_{IB})) \\ &= \exp(\log(\Phi_{AB})) \\ &= \Phi_{AB} \\ &= \exp({}_A\boldsymbol{\varphi}_{AB}) = \exp({}_B\boldsymbol{\varphi}_{AB}) \end{aligned} \tag{25}$$