Primer on Spatial Algebra

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Contents

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Table 1: Notation

Symbol	Dimension	Meaning
$\mathcal{B}$		Coordinate frame name (always capital calligraphic letters). The corresponding origin is denoted with B
B		Body name or center of frame $\mathcal{B}$
${}_{\mathcal{C}}\boldsymbol{r}_{AB}$		Position vector from point A to point B represented in $\mathcal{C}$ frame
$\hat{\boldsymbol{v}}$		Spatial velocity (abstract vector, no coordinate representation yet)
$\hat{\boldsymbol{a}}$		Spatial acceleration (abstract vector, no coordinate representation yet)
$\hat{\boldsymbol{I}}$		Spatial inertia (mapping from spatial velocity to spatial momentum)
${}_{\mathcal{O}}\hat{\boldsymbol{v}}_B$	6×1	Spatial velocity of body B w.r.t. $\mathcal{O}$'s origin as reference point and coordinates represented in frame $\mathcal{O}$
${}_{\mathcal{O}}\hat{\boldsymbol{v}}_B^{\times}$	6×6	Cross product matrix of above velocity
${}_{\mathcal{O}}\boldsymbol{I}$	3×3	Inertia matrix
${}_{\mathcal{O}}\hat{\boldsymbol{I}}$	6×6	Spatial inertia represented in frame $\mathcal{O}$
$\times$		Spatial vector cross product for motion terms or 'normal' cross product for 3-dim vectors (clear from context)
$\times^*$		Spatial vector cross product for force terms
M^6		Vector space of motion spatial coordinates
F^6		Vector space of force spatial coordinates

1 Introduction

Purpose of this document is to provide an overview on spatial velocities, or as Roy Featherstone says "The Easy Way to do Rigid Body Dynamics." This tutorial is largely based on workshop slides[2] and the book 'Rigid Body Dynamics' [1] by Featherstone. It might also be helpful to look at the tutorials [2, 3]. In Table 1, we summarize the notation used throughout this document.

We will frequently use the cross product matrix (skew-symmetric matrix) of a 3 dimensional vector, which is defined as

$$\boldsymbol{r}^{\times} = \begin{bmatrix} r_x \\ r_y \\ r_z \end{bmatrix}^{\times} = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}. \tag{1}$$

2 Spatial vectors

Spatial vectors are objects, represented by a tuple of length six, that represent a linear and angular quantity such as force and torque, or linear and angular velocity. All spatial vectors in this document have a hat on top.

[2] `http://royfeatherstone.org/spatial/slides.pdf`

2.1 Spatial velocity and acceleration

A spatial velocity describes the motion of a rigid body as a whole, not the motion of an individual point of that body. The spatial velocity of a rigid body B is denoted

$$ {}_{\mathcal{O}}\hat{\boldsymbol{v}}_B = {}^{\mathcal{O}}_{\mathcal{O}}\hat{\boldsymbol{v}}_B = \begin{bmatrix} \omega_x & \omega_y & \omega_z & v_{\mathcal{O}x} & v_{\mathcal{O}y} & v_{\mathcal{O}z} \end{bmatrix}^\top = \begin{bmatrix} {}^{\mathcal{O}}\boldsymbol{\omega} \\ {}^{\mathcal{O}}_{\mathcal{O}}\boldsymbol{v} \end{bmatrix} . \tag{2} $$

This notation means: It is B's spatial velocity $\hat{\boldsymbol{v}}$ w.r.t. the reference point $\mathcal{O}$ and the coordinates are represented according to the frame axes $\mathcal{O}$. In the following, we drop one of the indices on the left of a spatial vector by assuming the reference point is always the origin of the reference frame.

In other words, the spatial velocity of a body w.r.t. a reference frame $\mathcal{O}$ is composed of two parts: a) The classical absolute angular velocity of the body and b) the absolute linear velocity of the body *induced* at the origin $\mathcal{O}$. For the latter, imagine the body extends infinitely in space and then write an expression for the velocity of the point on the body that coincides with $\mathcal{O}$ momentarily.

If the absolute linear velocity of a body w.r.t. the stationary frame $\mathcal{I}$ is given as $\boldsymbol{v}_{IB}$ (meaning the reference point on body B is moving with that velocity) and the absolute angular velocity is $\boldsymbol{\omega}_{IB}$, then the spatial velocity can easily be calculated with

$$ {}_{\mathcal{I}}\hat{\boldsymbol{v}}_B = \begin{bmatrix} {}_{\mathcal{I}}\boldsymbol{\omega}_{IB} \\ {}_{\mathcal{I}}\boldsymbol{v}_{IB} + {}_{\mathcal{I}}\boldsymbol{r}_{IB} \times {}_{\mathcal{I}}\boldsymbol{\omega}_{IB} \end{bmatrix} , \tag{3} $$

$$ {}_{\mathcal{B}}\hat{\boldsymbol{v}}_B = \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{\omega}_{IB} \\ {}_{\mathcal{B}}\boldsymbol{v}_{IB} \end{bmatrix} . \tag{4} $$

The relative velocity between two bodies is $\hat{\boldsymbol{v}}_{\text{rel}} = \hat{\boldsymbol{v}}_{12} = \hat{\boldsymbol{v}}_2 - \hat{\boldsymbol{v}}_1$. To make this subtraction component-wise, both the reference point and frame must be identical for all three terms. Two rigidly connected bodies have the same spatial velocity, i.e., zero relative velocity and zero relative acceleration.

Moving reference frames: If the reference coordinate system $\mathcal{O}$ happens to be moving (i.e., its orientation changes over time and/or the origin has a nonzero linear velocity), the spatial velocity ${}_{\mathcal{O}}\hat{\boldsymbol{v}}_B$ is *still* an absolute spatial velocity of the body w.r.t. a stationary reference frame that momentarily coincides with $\mathcal{O}$. This means

- The velocity ${}_{\mathcal{O}}\hat{\boldsymbol{v}}_{\mathcal{O}}$ of the moving frame expressed w.r.t. itself is not zero. It describes the velocity of frame $\mathcal{O}$ w.r.t. a stationary frame that momentarily concides with $\mathcal{O}$.
- The spatial velocity of a body w.r.t. a moving frame is *not* relative to the frame's movement. This is not to be confused with relative spatial velocities which are a difference between two absolute spatial velocities.

Time derivative: The general differentiation rule for spatial vectors ${}_{\mathcal{O}}\hat{\boldsymbol{s}}$ is

$$ {}_{\mathcal{O}}\left[\frac{\mathrm{d}}{\mathrm{dt}}\hat{\boldsymbol{s}}\right] = \underbrace{\frac{\mathrm{d}}{\mathrm{dt}}\left({}_{\mathcal{O}}\hat{\boldsymbol{s}}\right)}_{\text{componentwise}} + \underbrace{{}_{\mathcal{O}}\hat{\boldsymbol{v}}_{\mathcal{O}}}_{\substack{\text{vel of} \\ \text{coord. frame}}} \times {}_{\mathcal{O}}\hat{\boldsymbol{s}} . \tag{5} $$

For force terms (e.g., spatial force), the force cross product ($\times^*$) must be used (see Sect.2.3). The first term in the formula above accounts for the change of the vector in a stationary frame while the cross product adds the effect of a moving reference frame.

Spatial acceleration Spatial acceleration is the rate of change of spatial velocity, i.e., how much the spatial velocity vector changes over time.

$$ {}_{\mathcal{O}}\hat{\boldsymbol{a}} = {}_{\mathcal{O}}\left[\frac{\mathrm{d}}{\mathrm{dt}}\hat{\boldsymbol{v}}\right] = \begin{bmatrix} \dot{\boldsymbol{\omega}} \\ {}_{\mathcal{O}}\dot{\boldsymbol{v}} \end{bmatrix} . \tag{6} $$

Be aware that the linear component is not the acceleration of any point on the body because we are still considering how the 'flow' changes through the point $\mathcal{O}$ (which is *not* fixed to the body but fixed in space). Spatial acceleration is a true vector. We can perform calculations like

$$ \underbrace{\hat{\boldsymbol{a}}_B}_{\text{acc. body B}} = \underbrace{\hat{\boldsymbol{a}}_A}_{\text{acc. body A}} + \underbrace{\hat{\boldsymbol{a}}_{AB}}_{\text{acc. of B w.r.t. A}} , \tag{7} $$

and avoid any Coriolis terms! To numerically perform the addition, the reference frame must be identical for all quantities. Spatial accelerations are also true vectors, so it holds that $\hat{\boldsymbol{a}}_{\mathrm{rel}} = \hat{\boldsymbol{a}}_{12} = \hat{\boldsymbol{a}}_2 - \hat{\boldsymbol{a}}_1$.

Relative Velocities are of particular interest to describe systems of rigid bodies (like a robot), where the links are connected via joints. For example, the relative velocity of two bodies connected by a rotational/translational joint with axis $\hat{\boldsymbol{s}}$ and joint rate $\dot{q}$ is called joint velocity $\hat{\boldsymbol{v}}_J = \hat{\boldsymbol{s}}_J\dot{q}$. This quantity is conveniently expressed in the parent $\mathcal{P}$ or child $\mathcal{C}$ frame of the joint because the representations ${}_{\mathcal{C}}\hat{\boldsymbol{s}}_J$ or ${}_{\mathcal{P}}\hat{\boldsymbol{s}}_J$ are constant. When differentiating this quantity, we get the joint acceleration.

2.2 Spatial force

The spatial force

$$ {}_{\mathcal{O}}\hat{\boldsymbol{f}} = {}^{\mathcal{O}}_{\mathcal{O}}\hat{\boldsymbol{f}} = \begin{bmatrix} n_{\mathcal{O}x} & n_{\mathcal{O}y} & n_{\mathcal{O}z} & f_x & f_y & f_z \end{bmatrix}^\top = \begin{bmatrix} {}^{\mathcal{O}}_{\mathcal{O}}\boldsymbol{n} \\ {}^{\mathcal{O}}\boldsymbol{f} \end{bmatrix} , \tag{8} $$

consists of a linear force $\boldsymbol{f}$ acting along a line that passes through $\mathcal{O}$ and a couple $\boldsymbol{n}$[3] representing the total moment about $\mathcal{O}$. Again, we typically drop one of the prescripts. The total spatial force on a body is the sum of the individual spatial forces $\hat{\boldsymbol{f}}_{\mathrm{total}} = \sum \hat{\boldsymbol{f}}_i$. The sum can be computed component-wise if all forces are represented in the same reference frame. Action and reaction forces between two interacting bodies are $\hat{\boldsymbol{f}}_{\mathrm{action}} = -\hat{\boldsymbol{f}}_{\mathrm{reaction}}$, just like Newton's third law. The power delivered by a spatial force on a body is $\hat{\boldsymbol{f}} \cdot \hat{\boldsymbol{v}}$. The usual definition dot product works if reference point and frame of both quantities are the same. There is no inner product defined on M^6 or F^6, just between them.

2.3 Spatial Cross Products

Spatial cross products are defined such that the differentiation formula (5) looks like the Euler differentiation rule. The cross product is different for motion and force spatial vectors:
Motion cross motion

$$ {}_{\mathcal{O}}\hat{\boldsymbol{v}} \times {}_{\mathcal{O}}\hat{\boldsymbol{m}} = \begin{bmatrix} \boldsymbol{\omega} \\ {}_{\mathcal{O}}\boldsymbol{v} \end{bmatrix} \times \begin{bmatrix} \boldsymbol{m} \\ {}_{\mathcal{O}}\boldsymbol{m} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega} \times \boldsymbol{m} \\ \boldsymbol{\omega} \times {}_{\mathcal{O}}\boldsymbol{m} + {}_{\mathcal{O}}\boldsymbol{v} \times \boldsymbol{m} \end{bmatrix} . \tag{9} $$

[3]German: Freies Moment

Motion cross force

$$ {}_{\mathcal{O}}\hat{\boldsymbol{v}} \times^{*} {}_{\mathcal{O}}\hat{\boldsymbol{f}} = \begin{bmatrix} \boldsymbol{\omega} \\ {}_{\mathcal{O}}\boldsymbol{v} \end{bmatrix} \times^{*} \begin{bmatrix} {}_{\mathcal{O}}\boldsymbol{\tau} \\ \boldsymbol{f} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\omega} \times {}_{\mathcal{O}}\boldsymbol{\tau} + {}_{\mathcal{O}}\boldsymbol{v} \times \boldsymbol{f} \\ \boldsymbol{\omega} \times \boldsymbol{f} \end{bmatrix} . \tag{10} $$

Many properties of the cross product are found in [1].

2.4 Transforms

Transforming a spatial motion vector (velocity, acceleration) to another reference frame using the transform matrix

$$ {}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{A}} = \begin{bmatrix} \boldsymbol{R}_{\mathcal{BA}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{R}_{\mathcal{BA}} \end{bmatrix} \begin{bmatrix} \mathbb{I} & \boldsymbol{0} \\ -{}_{\mathcal{A}}\boldsymbol{r}^{\times}_{\mathcal{AB}} & \mathbb{I} \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_{\mathcal{BA}} & \boldsymbol{0} \\ -\boldsymbol{R}_{\mathcal{BA}}\, {}_{\mathcal{A}}\boldsymbol{r}^{\times}_{\mathcal{AB}} & \boldsymbol{R}_{\mathcal{BA}} \end{bmatrix} . \tag{11} $$

With this definition it holds that

$$ {}_{\mathcal{B}}\hat{\boldsymbol{m}} = {}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{A}}\ {}_{\mathcal{A}}\hat{\boldsymbol{m}} . \tag{12} $$

Our notation writes rotation matrices as $\boldsymbol{R}_{\mathcal{AB}}$ which means this rotation matrix transforms vectors represented in $\mathcal{B}$ frame into $\mathcal{A}$ frame (passive rotation). The inverse of the transform is given by

$$ {}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{A}}^{-1} = {}^{\mathcal{A}}\boldsymbol{X}_{\mathcal{B}} = \begin{bmatrix} \boldsymbol{R}^{\top}_{\mathcal{BA}} & \boldsymbol{0} \\ {}_{\mathcal{A}}\boldsymbol{r}^{\times}_{\mathcal{AB}}\, \boldsymbol{R}^{\top}_{\mathcal{BA}} & \boldsymbol{R}^{\top}_{\mathcal{BA}} \end{bmatrix} . \tag{13} $$

For force vectors, the star version must be used

$$ {}^{\mathcal{B}}\boldsymbol{X}^{*}_{\mathcal{A}} = ({}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{A}})^{-\top} , \tag{14} $$

$$ {}_{\mathcal{B}}\hat{\boldsymbol{f}} = {}^{\mathcal{B}}\boldsymbol{X}^{*}_{\mathcal{A}}\ {}_{\mathcal{A}}\hat{\boldsymbol{f}} . \tag{15} $$

The time derivative of the transform is given by

$$ \frac{\mathrm{d}}{\mathrm{dt}} {}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{A}} = {}_{\mathcal{B}}(\hat{\boldsymbol{v}}_{\mathcal{A}} - \hat{\boldsymbol{v}}_{\mathcal{B}}) \times\ {}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{A}} . \tag{16} $$

3 Spatial inertia

The rigid body spatial inertia tensor w.r.t. frame $\mathcal{O}$ is formed as

$$ {}_{\mathcal{O}}\hat{\boldsymbol{I}} = \begin{bmatrix} {}_{\mathcal{O}}\boldsymbol{I}_{\mathcal{O}} & m\, {}_{\mathcal{O}}\boldsymbol{r}^{\times}_{\mathcal{OC}} \\ m\, {}_{\mathcal{O}}\boldsymbol{r}^{\times\,\top}_{\mathcal{OC}} & m\mathbb{I} \end{bmatrix} , \tag{17} $$

with

$$ {}_{\mathcal{O}}\boldsymbol{I}_{\mathcal{O}} = {}_{\mathcal{O}}\boldsymbol{I}_{\mathcal{C}} + m\, {}_{\mathcal{O}}\boldsymbol{r}^{\times}_{\mathcal{OC}}\, {}_{\mathcal{O}}\boldsymbol{r}^{\times\,\top}_{\mathcal{OC}} , \tag{18} $$

where $\mathcal{C}$ denotes the center of mass of the body and ${}_{\mathcal{O}}\boldsymbol{I}_{\mathcal{C}}$ the classical 3×3 inertia matrix at the center of mass expressed in the frame $\mathcal{O}$. Typically, the frame $\mathcal{O}$ is rigidly attached to the body such that the representation of the tensor remains constant.

Two bodies that are rigidly joined together have a new inertia

$$ {}_{\mathcal{O}}\hat{\boldsymbol{I}}_{\text{combined}} = {}_{\mathcal{O}}\hat{\boldsymbol{I}}_1 + {}_{\mathcal{O}}\hat{\boldsymbol{I}}_2 . \tag{19} $$

There are no Steiner terms appearing and no need to compute the new center of mass. Both inertias must be w.r.t. the same frame $\mathcal{O}$.

To change the coordinate frame (i.e., both the reference and the origin), the formula is

$$ {}_{\mathcal{A}}\hat{\boldsymbol{I}} = {}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{A}}^{\top}\, {}_{\mathcal{B}}\hat{\boldsymbol{I}}\, {}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{A}} \,. \tag{20} $$

The time derivative of the spatial inertia of a body B represented in a body-attached coordinate frame $\mathcal{B}$ is

$$ {}_{\mathcal{B}}\left[\frac{\mathrm{d}}{\mathrm{dt}}\hat{\boldsymbol{I}}_B\right] = {}_{\mathcal{B}}\hat{\boldsymbol{v}}_B \times^{*} {}_{\mathcal{B}}\hat{\boldsymbol{I}}_B - {}_{\mathcal{B}}\hat{\boldsymbol{I}}_B\, {}_{\mathcal{B}}\hat{\boldsymbol{v}}_B^{\times} \tag{21} $$

This quantity is in general nonzero because of the motion of frame $\mathcal{B}$.

4 Motion of a single rigid body

The spatial momentum of a body is $\hat{\boldsymbol{h}} = \hat{\boldsymbol{I}}\hat{\boldsymbol{v}}$. The equation of motion of a single rigid body B in 3D space can be written in spatial notation w.r.t. a body-fixed coordinate frame $\mathcal{B}$ as

$$ \begin{aligned} {}_{\mathcal{B}}\hat{\boldsymbol{f}} &= \frac{\mathrm{d}}{\mathrm{dt}}\left({}_{\mathcal{B}}\hat{\boldsymbol{I}}_B\, {}_{\mathcal{B}}\hat{\boldsymbol{v}}_B\right) \\ &= {}_{\mathcal{B}}\hat{\boldsymbol{I}}_B\, {}_{\mathcal{B}}\hat{\boldsymbol{a}}_B + {}_{\mathcal{B}}\hat{\boldsymbol{v}}_B \times^{*} {}_{\mathcal{B}}\hat{\boldsymbol{I}}_B\, {}_{\mathcal{B}}\hat{\boldsymbol{v}}_B \,, \end{aligned} \tag{22} $$

in other words: Net spatial force equals rate of change of spatial momentum.

Example Working out these equations in detail recovers familiar formulas for the motion of a single body in classical 3D vector notation. We assume that the body-fixed frame has its origin at the center of mass.

$$ {}_{\mathcal{B}}\hat{\boldsymbol{I}}_B = \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{I}_B & \boldsymbol{0} \\ \boldsymbol{0} & m\mathbb{I} \end{bmatrix} \tag{23} $$

$$ {}_{\mathcal{B}}\hat{\boldsymbol{v}}_B = \begin{bmatrix} {}_{\mathcal{B}}\boldsymbol{\omega}_{IB} \\ {}_{\mathcal{B}}\boldsymbol{v}_B \end{bmatrix} \tag{24} $$

$$ {}_{\mathcal{B}}\hat{\boldsymbol{a}}_B = \begin{bmatrix} {}_{\mathcal{B}}\dot{\boldsymbol{\omega}}_{IB} \\ {}_{\mathcal{B}}\boldsymbol{a}_B \end{bmatrix} \tag{25} $$

$$ {}_{\mathcal{B}}\hat{\boldsymbol{f}} = \sum_i ({}^{\mathcal{B}}\boldsymbol{X}_i^{*}\, {}_i\hat{\boldsymbol{f}}_i) + {}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{I}} \begin{bmatrix} \boldsymbol{0} \\ m\, {}_{\mathcal{I}}\boldsymbol{g} \end{bmatrix} \tag{26} $$

Now assemble equation of motion (22) with external forces $\hat{\boldsymbol{f}}_i$:

$$ \sum_i ({}^{\mathcal{B}}\boldsymbol{X}_i^{*}\, {}_i\hat{\boldsymbol{f}}_i) + {}^{\mathcal{B}}\boldsymbol{X}_{\mathcal{I}} \begin{bmatrix} \boldsymbol{0} \\ m\, {}_{\mathcal{I}}\boldsymbol{g} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_B\, {}_{\mathcal{B}}\dot{\boldsymbol{\omega}}_{IB} + {}_{\mathcal{B}}\boldsymbol{\omega}_{IB} \times \boldsymbol{I}_B\, {}_{\mathcal{B}}\boldsymbol{\omega}_{IB} \\ m\, {}_{\mathcal{B}}\boldsymbol{a}_B + {}_{\mathcal{B}}\boldsymbol{\omega}_{IB} \times m\, {}_{\mathcal{B}}\boldsymbol{v}_B \end{bmatrix} \tag{27} $$

This is consistent with [4].

5 Recursive Newton Euler

To calculate the equations of motion (in particular inverse dynamics), one can employ the recursive Newton Euler algorithm. It consists of two passes along the robot's branches: First, velocities and accelerations are propagated from the robot's base to the end-effector. Second, the total spatial force necessary for the calculated acceleration for each body is found via (22) and projected onto the joint axis.

The procedure is most conveniently implemented when all spatial quantities are expressed in body-fixed frames.

Gravity is modeled by adding a fictitious acceleration to the root link. Alternatively, one could introduce the gravity force as an external force on all bodies.

There are different ways to model a floating-base robot. Conceptually the easiest (and typically implemented in libraries like RBDL[4]) is to define the zeroth body as the world link and introduce a 6 DoF joint between world link and robot base.

A few remarks on notation and convention:

1. The parent of body i is denoted $\lambda(i)$

2. The children of body i are denoted $\mu(i)$

3. Joint i connects between body $\lambda(i)$ and link i.

4. The joint velocity across a joint is the relative velocity of child w.r.t. parent, i.e.,

$$\hat{\boldsymbol{v}}_{Ji} = \hat{\boldsymbol{v}}_i - \hat{\boldsymbol{v}}_{\lambda(i)} \ . \tag{28}$$

   The same rule applies to joint accelerations.

5. The joint velocity can be expressed as a linear function of a subset of the generalized velocities pertaining to that joint:

$$\hat{\boldsymbol{v}}_{Ji} = \hat{\boldsymbol{S}}_{Ji}\dot{\boldsymbol{q}}_{Ji} \tag{29}$$

   For single-axis joints, $\boldsymbol{q}_{Ji}$ is a scalar and $\hat{\boldsymbol{S}}_{Ji}$ a column vector. For a 6 DoF joint emulating the floating base, $\hat{\boldsymbol{S}}$ is typically the identity matrix and $\boldsymbol{q}_{Ji}$ the floating base spatial velocity in base frame.

6. The joint force $\hat{\boldsymbol{f}}_{Ji}$ acts on body i whereas the negative force (reaction force) is felt by body $\lambda(i)$.

We summarize the algorithm in the following, which closely resembles [1, Table 5.1], but including the full frame indices:

1. Initialize zeroth link (world or fixed base link) with virtual gravity acceleration. If the z-axis is pointing upwards in frame $\mathcal{O}$, then ${}_{\mathcal{O}}\hat{\boldsymbol{a}}_g = [\ 0 \ \ 0 \ \ 0 \ \ 0 \ \ 0 \ \ g\ ]^\top$ with $g = 9.81\,\mathrm{m/s^2}$:

$$ {}_0\hat{\boldsymbol{v}}_0 = \hat{\boldsymbol{0}} \tag{30}$$

$$ {}_0\hat{\boldsymbol{a}}_0 = -\hat{\boldsymbol{a}}_g \tag{31}$$

2. For each consecutive link, the velocity and accelerations are calculated by adding the joint velocity and acceleration, respectively, to the parent's ones. A couple of helper quantities are defined to keep the equations simpler.

$$ {}_i\hat{\boldsymbol{v}}_{Ji} = {}_i\hat{\boldsymbol{S}}_{Ji}\ \dot{\boldsymbol{q}}_{Ji} \qquad \text{(joint velocity)} \tag{32}$$

$$ {}_i\hat{\boldsymbol{a}}_{Ji} = {}_i\hat{\boldsymbol{S}}_{Ji}\ \ddot{\boldsymbol{q}}_{Ji} + \frac{\mathrm{d}}{\mathrm{dt}}\left({}_i\hat{\boldsymbol{S}}_{Ji}\right)\ \dot{\boldsymbol{q}}_{Ji} + {}_i\hat{\boldsymbol{v}}_i \times {}_i\hat{\boldsymbol{v}}_{Ji} \qquad \text{(joint acceleration)} \tag{33}$$

$$ {}_i\hat{\boldsymbol{v}}_i = {}^i\boldsymbol{X}_{\lambda(i)}\ {}_{\lambda(i)}\hat{\boldsymbol{v}}_{\lambda(i)} + {}_i\hat{\boldsymbol{v}}_{Ji} \qquad \text{(link velocity)} \tag{34}$$

$$ {}_i\hat{\boldsymbol{a}}_i = {}^i\boldsymbol{X}_{\lambda(i)}\ {}_{\lambda(i)}\hat{\boldsymbol{a}}_{\lambda(i)} + {}_i\hat{\boldsymbol{a}}_{Ji} \qquad \text{(link acceleration)} \tag{35}$$

[4] `https://bitbucket.org/rbdl/rbdl`

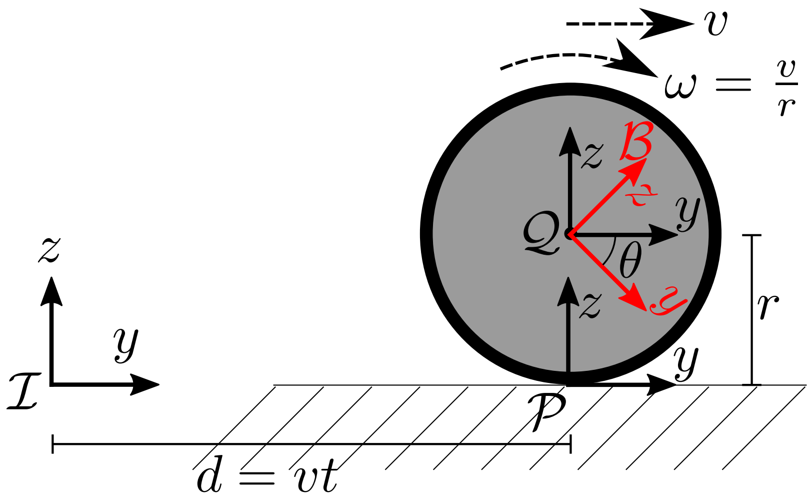

Figure 1: 2D ball rolling with velocity v without slippage on a plane. The inertial coordinate frame $\mathcal{I}$ is fixed while $\mathcal{P}$ and $\mathcal{Q}$ translate with the ball's center. The body-fixed frame $\mathcal{B}$ additionally rotates with the ball.

3. Now that the absolute accelerations are known for each link, one can employ (22) to calculate the net force that must act on each link to achieve that acceleration:

$$ {}_i\hat{\boldsymbol{f}}_i^{\text{net}} = {}_i\hat{\boldsymbol{I}}_i \; {}_i\hat{\boldsymbol{a}}_i + {}_i\hat{\boldsymbol{v}}_i \times^* {}_i\hat{\boldsymbol{I}}_i \; {}_i\hat{\boldsymbol{v}}_i \; . \tag{36} $$

4. The net force is a sum of the external forces acting on the body and all forces transmitted through joints connected to that body. The force transmitted through the parent joint must therefore be

$$ {}_i\hat{\boldsymbol{f}}_{Ji} = {}_i\hat{\boldsymbol{f}}_i^{\text{net}} - {}^i\boldsymbol{X}_0^* \; {}_0\hat{\boldsymbol{f}}_i^{\text{ext}} + \sum_{j \in \mu(i)} {}^i\boldsymbol{X}_j^* \; {}_j\hat{\boldsymbol{f}}_{Jj} \; . \tag{37} $$

   This can be computed recursively for each body from the end-effector inwards.

5. Finally, the torque/force transmitted for each of the joint axes can be computed with

$$ \boldsymbol{\tau}_{Ji} = {}_i\hat{\boldsymbol{S}}_{Ji}^\top \; {}_i\hat{\boldsymbol{f}}_{Ji} \; . \tag{38} $$

6 Examples

6.1 Rolling ball on a plane

We consider a planar ball B with radius r that rolls without slippage on flat ground with velocity v. Figure 1 displays a schematic drawing with the relevant modeling quantities. We assume that the motion starts in a zero configuration, i.e.,

$$ d = vt \; , \tag{39} $$

$$ \theta = \omega t \; . \tag{40} $$

Some basic quantities are defined in the following:

$$
{}_{\mathcal{I}}\boldsymbol{r}_{IP} = \begin{bmatrix} 0 & vt & 0 \end{bmatrix}^{\top}, \tag{41}
$$

$$
{}_{\mathcal{P}}\boldsymbol{r}_{PQ} = \begin{bmatrix} 0 & 0 & r \end{bmatrix}^{\top}, \tag{42}
$$

$$
\boldsymbol{R}_{\mathcal{QB}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{43}
$$

Spatial Transforms The spatial transforms are in general configuration dependent. For this example we have for example

$$
{}^{\mathcal{P}}\boldsymbol{X}_{\mathcal{I}} = \begin{bmatrix} \mathbb{I} & \boldsymbol{0} \\ -{}_{\mathcal{I}}\boldsymbol{r}_{IP}^{\times} & \mathbb{I} \end{bmatrix} \tag{44}
$$

Spatial Velocities By inspection, the spatial velocity of the ball with respect to coordinate system $\mathcal{P}$ is

$$
{}_{\mathcal{P}}\hat{\boldsymbol{v}}_B = \begin{bmatrix} -\frac{v}{r} & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\top}, \tag{45}
$$

because the rolling motion leads to instantaneous rotation around point P. On the other hand, the spatial velocity in frame $\mathcal{I}$ additionally has a nonzero component in the linear z direction because the rolling motion induces an upwards velocity at the point I, hence

$$
{}_{\mathcal{I}}\hat{\boldsymbol{v}}_B = \begin{bmatrix} -\frac{v}{r} & 0 & 0 & 0 & 0 & \frac{v}{r}vt \end{bmatrix}^{\top} = {}^{\mathcal{I}}\boldsymbol{X}_{\mathcal{P}}\ {}_{\mathcal{P}}\hat{\boldsymbol{v}}_B . \tag{46}
$$

With respect to coordinate system $\mathcal{Q}$ we get

$$
{}_{\mathcal{Q}}\hat{\boldsymbol{v}}_B = \begin{bmatrix} -\frac{v}{r} & 0 & 0 & v & 0 & 0 \end{bmatrix}^{\top}. \tag{47}
$$

Care must be taken if we consider system $\mathcal{B}$, because the forward velocity at point B must be projected onto the instantaneous coordinate axes, i.e.,

$$
{}_{\mathcal{B}}\hat{\boldsymbol{v}}_B = \begin{bmatrix} -\frac{v}{r} \\ 0 \\ 0 \\ \boldsymbol{R}_{\mathcal{BQ}} \begin{bmatrix} 0 \\ v \\ 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} -\frac{v}{r} \\ 0 \\ 0 \\ 0 \\ v\cos\theta \\ v\sin\theta \end{bmatrix}. \tag{48}
$$

Spatial acceleration We show how to calculate the spatial acceleration in two ways

1. Directly differentiate ${}_{\mathcal{I}}\hat{\boldsymbol{v}}_B$ (since it is represented in a stationary frame).

$$
{}_{\mathcal{I}}\hat{\boldsymbol{a}}_B = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{v^2}{r} \end{bmatrix}^{\top} \tag{49}
$$

2. Use the differentiation rule on ${}_{\mathcal{P}}\hat{\boldsymbol{v}}_B$ and then transform it to the $\mathcal{I}$ frame for comparison.

$$
{}_{\mathcal{P}}\hat{\boldsymbol{a}}_B = 0 + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ v \\ 0 \end{bmatrix} \times \begin{bmatrix} -\frac{v}{r} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{v^2}{r} \end{bmatrix} \tag{50}
$$

$$
{}_{\mathcal{I}}\hat{\boldsymbol{a}}_B = {}^{\mathcal{I}}\boldsymbol{X}_{\mathcal{P}}\ {}_{\mathcal{P}}\hat{\boldsymbol{a}}_B = \begin{bmatrix} \mathbb{I} & \mathbf{0} \\ -{}_{\mathcal{P}}\boldsymbol{r}_{PI}^{\times} & \mathbb{I} \end{bmatrix} {}_{\mathcal{P}}\hat{\boldsymbol{a}}_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \frac{v^2}{r} \end{bmatrix} \tag{51}
$$

Both methods generate consistent results. Notice that the spatial acceleration of the rolling sphere is upwards. This is not the acceleration of any point on the body and may seem rather unintuitive initially.

6.2 Floating base robot with a single link attached

coming soon ...

References

[1] R. Featherstone, *Rigid body dynamics algorithms.* Springer, 2014.

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